LMI Approach for Stability of Neural Networks

K. Ratchagit

Department of Mathematics
Faculty of Science, Maejo University
Chiang Mai 50290, Thailand
e-mail: kreangkri@mju.ac.th

***Abstract*:** In this paper, we derive a sufficient condition for asymptotic stability of the zero solution of delay-difference control system of Hopfield neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method.

***Keywords*:** Asymptotic stability, Hopfield neural networks, lyapunov function, delay-difference control system, matrix inequalities.

1. Introduction

In recent decades, Hopfield neural networks have been extensively studied in many aspects and successfully applied to many fields such as pattern identifying, voice recognizing, system controlling, signal processing systems, static image treatment, and solving nonlinear algebraic system, etc. Such applications are based on the existence of equilibrium points, and qualitative properties of systems. In electronic implementation, time delays occur due to some reasons such as circuit integration, switching delays of the amplifiers and communication delays, etc. Therefore, the study of the asymptotic stability of Hopfield neural networks with delays is of particular importance to manufacturing high quality microelectronic Hopfield neural networks.

While stability analysis of continuous-time neural networks can employ the stability theory of differential system (Liu *et al.* 2003), it is much harder to study the stability of discrete-time neural networks (Elaydi and Peterson 1990) with time delays (Arik 2005) or impulses (Gubta and Jin 1996). The techniques currently available in the literature for discrete-time systems are mostly based on the construction Lyapunov second method (Hale 1977). For Lyapunov second method, it is well known that no general rule exists to guide the construction of a proper Lyapunov function for a given system. In fact, the construction of the Lyapunov function becomes a very difficult task.

In this paper, we consider delay-difference control system of Hopfield neural networks of the form

$$v(k+1) = -Av(k) + BS(v(k-h)) + Cu(k) + f\,, \qquad (1)$$

where $v(k) \in \Omega \subseteq \mathbf{R}^n$ is the neuron state vector, $h \ge 0$, $A = diag\{a_1,\ldots,a_n\}$, $a_i \ge 0$, $i = 1,2,...,n$ is the $n \times n$ constant relaxation matrix, B is the $n \times n$ constant weight matrix, C is $n \times m$ constant matrix, $u(k) \in \mathbf{R}^m$ is the control vector, $f = (f_1,\ldots,f_n) \in \mathbf{R}^n$ is the constant external input vector and $S(z) = [s_1(z_1),\ldots,s_n(z_n)]^T$ with $s_i \in C^1\left[\mathbf{R},(-1,1)\right]$ where s_i is the neuron activations and monotonically increasing for each $i = 1,2,...,n$.

The asymptotic stability of the zero solution of the delay-differential system of Hopfield neural networks has been developed during the past several years. We refer to monographs by Burton (Burton 1993) and Ye (Ye 1944) and the references cited therein. Much less is known regarding the asymptotic stability of the zero solution of the delay-difference control system of Hopfield neural networks. Therefore, the purpose of this paper is to establish sufficient condition for the asymptotic stability of the zero solution of equation (1) in terms of certain matrix inequalities.

2. Preliminaries

The following notations will be used throughout the paper. $\mathbf{R}^+$ denotes the set of all non-negative real numbers; $\mathbf{Z}^+$ denotes the set of all non-negative integers; $\mathbf{R}^n$ denotes the n-finite-dimensional Euclidean space with the Euclidean norm $\|.\|$ and the scalar product between x and y is defined by $x^T y$; $\mathbf{R}^{n\times m}$ denotes the set of all $(n \times m)$-matrices; and A^T denotes the transpose of the matrix A; Matrix $Q \in \mathbf{R}^{n\times n}$ is positive semidefinite $(Q \ge 0)$ if $x^T Qx \ge 0$, for all $x \in \mathbf{R}^n$. If $x^T Qx > 0 (x^T Qx < 0$, resp.) for any $x \ne 0$, then Q is positive (negative, resp.) definite and denoted by $Q > 0, (Q < 0,$ resp.). It is easy to verify that $Q > 0$, $(Q < 0,$ resp.) iff $\exists \beta > 0$:
$x^T Qx \ge \beta \|x\|^2, \forall x \in \mathbf{R}^n$,
$(\exists \beta > 0: x^T Qx \le -\beta \|x\|^2, \forall x \in \mathbf{R}^n,$ resp.).

Lemma 2.1 (Hale 1977) The zero solution of difference system is asymptotic stability if there exists a positive definite function $V(x): \mathbf{R}^n \to \mathbf{R}^+$ such that

$$\exists \beta > 0 : \Delta V(x(k)) = V(x(k+1)) - V(x(k)) \le -\beta \|x(k)\|^2,$$

along the solution of the system. In case the above condition holds for all $x(k) \in V_\delta$, we say that the zero solution is locally asymptotically stable.

Lemma 2.2 For any constant symmetric matrix $M \in \mathbf{R}^{n\times n}$, $M = M^T > 0$, scalar $s \in \mathbf{Z}^+/\{0\}$, vector function $W:[0,s] \to \mathbf{R}^n$, we have

$$s\sum_{i=0}^{s-1}(w^T(i)Mw(i)) \ge \left(\sum_{i=0}^{s-1} w(i)\right)^T M \left(\sum_{i=0}^{s-1} w(i)\right).$$

We present the following technical lemmas, which will be used in the proof of our main result.

3. Main results

In this section, we consider the sufficient condition for asymptotic stability of the zero solution v^* of (1) in terms of certain matrix inequalities. Without loss of generality, we can assume that $v^* = 0, S(0) = 0$ and f =0 (for otherwise, we let $x = v - v^*$ and define

$$S(x) = S(x+v^*) - S(v^*)) .$$

The new form of equation (1) is now given by

$$x(k+1) = -Ax(k) + BS(x(k-h)) + Cu(k) . \qquad (2)$$

This is a basic requirement for controller design. Now, we are interested designing a feedback controller for the system equation (2) as

$$u(k) = Kx(k),$$

where K is $n \times m$ constant control gain matrix.

The new form of (2) is now given by

$$x(k+1) = -Ax(k) + BS(x(k-h)) + CKx(k) . \qquad (3)$$

Throughout this paper we assume the neuron activations $s_i(x_i)$, $i = 1,2,\ldots,n$ is bounded and monotonically nondecreasing on $\mathbf{R}$, and $s_i(x_i)$ is Lipschitz continuous, that is, there exist constant $l_i > 0, i = 1,2,\ldots,n$ such that

$$|s_i(r_1) - s_i(r_2)| \le l_i |r_1 - r_2|, \ \forall r_1, r_2 \in \mathbf{R}. \qquad (4)$$

By condition equation (4), $s_i(x_i)$ satisfy

$$|s_i(x_i)| \le l_i |x_i|, \ i = 1,2,...,n . \qquad (5)$$

Theorem 3.1 The zero solution of the delay-difference control system (3) is asymptotically stable if there exist symmetric positive definite matrices P,G,W and $L = diag[l_1,\ldots,l_n] > 0$ satisfying the following matrix inequalities of the form

$$\psi = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0, \qquad (6)$$

where

$$(1,1) = hG + W,$$

$$(2,2) = -W,$$

$$(3,3) = -hG.$$

Proof Consider the Lyapunov function

$V(y(k)) = V_1(y(k)) + V_2(y(k))$, where

$$V_1(y(k)) = \sum_{i=k-h}^{k-1}(h-k+i)x^T(i)Gx(i),$$

$$V_2(y(k)) = \sum_{i=k-h}^{k-1} x^T(i)Wx(i),$$

G and W being symmetric positive definite solutions of (6) and $y(k) = [x(k), x(k-h)]$.

Then difference of $V(y(k))$ along trajectory of solution of (3) is given by

$$\Delta V(y(k)) = \Delta V_1(y(k)) + \Delta V_2(y(k)),$$

where

$$\begin{aligned}\Delta V_1(y(k)) &= \Delta\left(\sum_{i=k-h}^{k-1}(h-k+i)x^T(i)Gx(i)\right) \\ &= hx^T(k)Gx(k) - \sum_{i=k-h+1}^{k} x^T(i)Gx(i), \\ \Delta V_2(y(k)) &= \Delta\left(\sum_{i=k-h}^{k-1} x^T(i)Wx(i)\right) \\ &= x^T(k)Wx(k) - x^T(k-h)Wx(k-h),\end{aligned} \qquad (7)$$

Then we have

$$\Delta V \le x^T(k)[hG+W]x(k) - x^T(k-h)Wx(k-h) - \sum_{i=k-h}^{k-1} x^T(i)Gx(i).$$

Using Lemma 2.2, we obtain

$$\sum_{i=k-h}^{k-1} x^T(i)Gx(i) \ge \left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)^T (hG)\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right).$$

From the above inequality it follows that:

$$\Delta V \le x^T(k)[hG+W]x(k) - x^T(k-h)Wx(k-h)$$
$$-\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)^T (hG)\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)$$

$$= \left(x^T(k), x^T(k-h), (\frac{1}{h}\sum_{i=k-h}^{k-1} x(i))^T \right)\begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix}$$

$$\times \begin{pmatrix} x(k) \\ x(k-h) \\ (\frac{1}{h}\sum_{i=k-h+1}^{k} x(i)) \end{pmatrix}$$

$$= y^T(k)\psi y(k)$$

where

$$(1,1)=hG+W,$$

$$(2,2)=-W,$$

$$(3,3)=-hG,$$

$$y(k)=\begin{pmatrix} x(k) \\ x(k-h) \\ (\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)) \end{pmatrix}.$$

By the condition (6), $\Delta V(y(k))$ is negative definite, namely there is a number $\beta > 0$ such that $\Delta V(y(k)) \le -\beta\|y(k)\|^2$, and hence, the asymptotic stability of the system immediately follows from Lemma 2.1. This completes the proof.

4. Conclusions

In this paper, based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of delay-difference control system of Hopfield neural networks in terms of certain matrix inequalities.

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References

[1] K. Abdelwahab and R.B. Guenther, An introduction to numerical methods a MATLAB approach, Chapman and Hall/CRC, New York, 2002.

[2] S. Barnett and R.G. Cameron, Introduction to Mathematical Control Theory, Oxford, Clarendon Press, 1985.

[3] J. Lu, G. Chen, A new chaotic attractor coined, Int. J. Bifurc. Chaos 12 (2002) 659-661.

[4] V.N. Phat, Constrained Control Problems of Discrete Processes, World Scientific Publisher, Singapore-NewJersey-London,1996.

[5] V.N. Phat, Introduction to Mathematical Control Theory, Hanoi National University Publisher, Hanoi, 2001.

[6] VN Phat, J. Jiang, A.V. Savkin and I. Petersen, Robust stabilization of linear uncertain discrete-time systems via a limited communication channel. Systems and Control Letters. 53(2004), 347-360 (SCI)

[7] VN Phat and J. Jiang, Feedback stabilization of nonlinear discrete-time systems via a digital communication channel. Int. J. of Math. and Math. Sci., 1(2005), 43-56.

[8] VN Phat, Robust stability and stabilizability of uncertain linear hybrid systems with state delays. IEEE Trans. on CAS II, 52(2005), 94-98 (SCI)

[9] VN Phat, N.M. Linh and T.D. Phuong, Sufficient conditions for strong stability of non-linear time-varying control systems with state delays. Acta Math. Vietnamica, 30(2005), 69-86.

[10] VN Phat and A.V. Savkin, Robust set-valued state estimation for linear uncertain systems in Hilbert spaces. Nonl. Func. Anal. Appl., 10(2005), 285-298.

[11] VN Phat and S. Pairote, Global stabilization of linear periodically time-varying switched systems via matrix inequalities. J. Control Theory Appl. 1(2006), 26-31.

[12] P. Niamsup and VN Phat, Stability of linear time-varying delay systems and applications to control problems, J. Comput. Appll. Math. 194(2006), 343-356 .

[13] VN Phat, Global stabilization for linear continuous time-varying systems Appl. Math. Comput. 175(2006), 1730-1743.

[14] VN Phat and P. Niamsup, Stabilization of linear non-autonomous systems with norm bounded controls. J. Optim. Theory Appl., 131(2006), 135-149.

[15] S. Pairote and VN Phat, Exponential stability of switched linear systems with time-varying delay, Elect. J. Diff. Equations, 59(2007), 1-10

[16] Q.P. Ha, H. Trinh and VN Phat, Design of Reduced-Order Observers for Global State Feedback Control of Multi-Agent Systems, Int. J. of Aut. Control, N2/3, 1(2007), 165-181.

[17] VN Phat and PT Nam, Exponential stability and stabilization of uncertain linear time-varying systems using parameter dependent Lyapunov function. Int. J. of Control, 80(2007), 1333-1341.

[18] P.T. Nam and VN Phat, Robust exponential stability and stabilization of linear uncertain polytopic time-delay systems. J. Control Theory Appl., 6(2008), 163-170

[19] P. Niamsup, K. Mukdasai and VN Phat, Linear uncertain non-autonomous time-delay systems: Stability and stabilizability via Riccati equations. Elect. J. Diff. Equations., 26(2008), 1-10.

[20] VN Phat, D.Q. Vinh and N. S. Bay, L2−stabilization and H∞ control for linear non-autonomous time-delay systems in Hilbert spaces via Riccati equations, Adv. in Nonl. Var. Ineq., 11(2008), 75-86.

[21] P. Niamsup and K. Mukdasai and VN Phat, Improved exponential stability for time- varying systems with nonlinear delayed perturbations, Appl. Math. Comput., 204(2008), 490-495.
[22] VN Phat and Q.P. Ha, New characterization of stabilizability via Riccati equations for LTV systems. IMA J. Math. Contr. Inform., 25(2008), 419-429.
[23] PT Nam and VN Phat, An improved stability criterion for a class of neutral deferential equations. Appl. Math. Letters, 22(2009), 31-35.
[24] VN Phat, T. Bormat and P. Niamsup, Switching design for exponential stability of a class of nonlinear hybrid time-delay systems, Nonlinear Analysis: Hybrid Systems, 3(2009), 1-10
[25] VN Phat and PT Nam, Robust stabilization of linear systems with delayed state and control, J. Optim. Theory Appl., 140(2009), 287-299.
[26] L.V. Hien, Q.P. Haand VN Phat, Stability and stabilization of switched linear dynamic systems with time delay and uncertainties Appl. Math. Comput, 210(2009), 223-231.
[27] VN Phat and LV Hien, An application of Razumikhin theorem to exponential stability for linear non-autonomous systems with arbitrary time-varying delays, Appl. Math. Letters, 22(2009), 1412-1417.
[28] LV Hien and VN Phat, Exponential stability and stabilization of a class of uncertain linear time-delay systems, J. of the Franklin Institute, 346(2009), 611-625.
[29] LV Hien and VN Phat, Delay feedback control in exponential stabilization of linear time-varying systems with input delay, IMA J. Math. Contr. Inform., 26(2009), 163-177.
[30] LV Hien and VN Phat, Exponential stabilization for a class of hybrid systems with mixed delays in state and control, Nonlinear Analysis: Hybrid Systems, 3(2009), 259-265.
[31] VN Phat, Memoryless $H\infty$ controller design for switched nonlinear systems with mixed time-varying delays, Int. J. of Control, 82(2009), 1889-1898,
[32] VN Phat and Q.P. Ha, $H\infty$ control and exponential stability for a class of nonlinear non-autonomous systems with time-varying delay, J. Optim. Theory Appl., 142(2009), 603-618.
[33] P. Niamsup and VN Phat, $H\infty$ optimal control of LTV systems with time-varying delay via controllability approach, ScienceAsia, 35(2009), 284-289.